TMA4170 Fourier Analysis

Spaces

Moderate decreasing functions f on $\mathbb{R}$:

(1) $\exists\, \varepsilon > 0,\ A > 0$ such that $|f(x)| \leq \dfrac{A}{(1+|x|^2)^{\frac{1+\varepsilon}{2}}}$ $\left(\leq \dfrac{A}{|x|^{1+\varepsilon}} \right)$

Rapidly decreasing functions $f \in C^\infty(\mathbb{R})$:

(2) $\forall\, k, l \in \mathbb{N}$, $\sup\limits_{x \in \mathbb{R}} |x|^k |f^{(l)}(x)| < \infty$

$C_m(\mathbb{R}) := \{ f \in C(\mathbb{R}) :$ (1) holds $\}$ Ex: $f(x) = \dfrac{1}{1+|x|^2}$

$S(\mathbb{R}) := \{ f \in C^\infty(\mathbb{R}) :$ (2) holds $\}$ Ex: $f(x) = e^{-x^2}$

Schwartz space

Integration in $\mathbb{R}$

$$\int_{-\infty}^{\infty} f(x)\,dx := \lim_{N\to\infty} \int_{-N}^{N} f(x)\,dx$$

Improper Riemann or Lebesgue integral

Exists for $f \in C_m(\mathbb{R})$ (and $f \in L^1(\mathbb{R})$)

Lem. 2: Properties in $C_m(\mathbb{R})$ (or $L^1(\mathbb{R})$)

(a) $\int_{-\infty}^{\infty} (a f(x) + b g(x)) = a \int_{-\infty}^{\infty} f + b \int_{-\infty}^{\infty} g$ (linearity)

(b) $\int_{-\infty}^{\infty} f(x-h)\,dx \overset{h\in\mathbb{R}}{=} \int_{-\infty}^{\infty} f(x)\,dx$ (translation invariance)

(c) $\delta \int_{-\infty}^{\infty} f(\delta x)\,dx \overset{\delta>0}{=} \int_{-\infty}^{\infty} f(x)\,dx$ (scaling property)

(d) $\int_{-\infty}^{\infty} |f(x-h) - f(x)|\,dx \xrightarrow[h\to 0]{} 0$ (continuity of translation)

Fouriertransform in $C_m(\mathbb{R})$

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx, \qquad \xi \in \mathbb{R}$$

Well-defined for $f \in C_m(\mathbb{R})$ (and $f \in L^1(\mathbb{R})$)

Remark:

(a) $\|f\|_\infty \leq \|f\|_1$

(b) $f \in C_m(\mathbb{R}) \Rightarrow \hat{f} \in C_0(\mathbb{R}) = \{ f \in C(\mathbb{R}) : f(x) \to 0 \text{ as } |x| \to \infty \}$

(c) BUT: $f \in C_m \not\Rightarrow \hat{f} \in C_m$! How to make sense of $\mathcal{F}^{-1}[\hat{f}]\ (= f)$?

Today: $f \in S(\mathbb{R}) \Rightarrow \hat{f} \in S(\mathbb{R})$ Work in $S(\mathbb{R})$